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ABSTRACT

Using the exact solution for a pure Dirac particle (g=2) in a constant, homogeneous and arbitrarily strong magnetic field, Heisenberg and Euler and Weisskopf derived an exact expression for the non-linear part, L_1 say, of the Lagrangian of the electromagnetic field.

When the anomalous magnetic moment of the electron ($g\neq 2$) is considered, by adding a phenomenononological Pauli term to the Dirac equation, the exact solutions can still be found. Using these exact solutions we derive, in addition to L_1, a new non-linear term, L_2 say, which is correct to relative order α^2.

An exact expression for the corrections to the Lagrangian of the electromagnetic field has been obtained by Heisenberg and Euler[1] and by Weisskopf[2] for fields $\underset{\sim}{F}(\underset{\sim}{E},\underset{\sim}{H})$ satisfying the conditions

$$\frac{\hbar}{mc}\,|\nabla \underset{\sim}{F}| \ll |\underset{\sim}{F}|\,, \qquad \frac{\hbar}{mc^2}\left|\frac{\partial \underset{\sim}{F}}{\partial t}\right| \ll |\underset{\sim}{F}|\,, \tag{1}$$

i.e. for slowly varying fields. In this analysis use was made of the exact solution for a pure Dirac particle (g=2) in a constant, homogeneous and arbitrarily strong magnetic field.[2,3] The derivation is also based on the assumption that the addition w' to the classical energy density of the electromagnetic field

$$w_0 = \frac{1}{8\pi}\,(E^2 + H^2) \tag{2}$$

due to the existence of the electron-positron vacuum, is exactly equal to the energy density of the vacuum electrons (i.e. the electrons which fill the negative energy sea postulated by Dirac) minus the potential energy of the electrons in the external electromagnetic field. This led to the conclusion that the addition L_1 to the classical Lagrangian

$$L_0 = \frac{1}{8\pi}\,(E^2 - H^2) \tag{3}$$

is equal to the negative of the total energy density of the electron-positron vacuum, w_m say, in the presence of an external field. Then, since L_1 is only a function of two independent invariants, E^2-H^2 and $(\mathbf{E}\cdot\mathbf{H})^2$, it is sufficient to obtain the value of w_m for particular field configurations. In particular, the magnetic field is taken to be constant and homogeneous along the z axis and the electrostatic field is chosen parallel to it. In this paper we are interested only in the corrections to the Lagrangian of the magnetic field and thus we take $\mathbf{E}=0$. We will return to the more general case of $\mathbf{E}\neq 0$ in a later publication.

The possible values of the energy of a vacuum electron (neglecting its anomalous magnetic moment) in a constant magnetic field H directed along the z axis are[2,3,4]

$$\mathcal{E}^{(0)}_{n,s}(p) \equiv \frac{E^{(0)}_{n,s}(p)}{mc^2} = -\sqrt{1+p^2+(2n+1+s)\frac{H}{H_c}} \quad , \qquad (4)$$

where

$$p \equiv \frac{P_z}{mc} \quad , \quad H_c = \frac{m^2c^3}{e\hbar} = 4\cdot 414\times 10^{13}\ \text{gauss} \quad ,$$

and where n = 0,1,2,... is the principal quantum number, $s = \pm 1$ is the spin variable, P_z is the momentum of the particle along the z axis and the superscript on $\mathcal{E}$ refers to the fact that the anomalous magnetic moment has been ignored. By use of Equation (4) it was found that[2)]

$$L_1 = \frac{m^4}{8\pi^2}\int_0^\infty \frac{d\eta}{\eta^3}\, e^{-\eta}\left\{\eta H^* \coth(\eta H^*) - 1 - \tfrac{1}{3}\eta^2 H^{*2}\right\}, \quad (5)$$

where

$$H^* = \frac{H}{H_c} \quad . \quad (6)$$

It follows that (where $\lambda_c = \frac{\hbar}{mc}$)

$$L_1 = \frac{1}{(4\pi)^2}\,\frac{2}{45}\,\frac{mc^2}{\lambda_c^3}\, H^{*4} \quad \text{for } H^* \ll 1 \quad , \quad (7)$$

and

$$L_1 = \frac{1}{24\pi^2}\,\frac{mc^2}{\lambda_c^3}\, H^{*2} \ln H^* \quad \text{for } H^* \gg 1 \quad . \quad (8)$$

which represent the weak and strong field limits[3)], respectively.

It is our purpose in this communication to obtain the corrections L_2 to $L_0 + L_1$ which result from the fact that the electron has an anomalous magnetic moment (AMM). Now the origin of this extra magnetic moment is explained by taking into account the fact that the electron can emit or absorb virtual photons and it can be taken care of in a phenomenological manner by adding the so-called Pauli anomalous moment interaction term ($\gamma_\mu \gamma_\nu F_{\mu\nu}$) to the usual Dirac Hamiltonian. In the case of a pure magnetic field the Dirac equation may thus be written

$$i\,\frac{\partial \psi}{\partial t} = \left\{ \underset{\sim}{\alpha}\cdot(\underset{\sim}{p} + e\underset{\sim}{A}) + \gamma_4 m + \mu\gamma_4\,\underset{\sim}{\Sigma}\cdot\underset{\sim}{H} \right\}\psi \quad , \tag{9}$$

where μ is the anomalous magnetic moment. Different values for the energy eigenvalues derived from the equation are quoted in the literature.[6,7] In view of this discrepancy, we have re-derived the result (See Appendix A) and we find, in agreement with reference 7, that the energy eigenvalues of the vacuum electrons are given by

$$\mathcal{E}_{n,s}(p) \equiv \frac{E_{n,s}(p)}{mc^2} = -\left\{ p^2 + \left[\left\{ 1 + (2n+s+1)H' \right\}^{1/2} + \tfrac{1}{2}\, s\, a\, H' \right]^2 \right\}^{1/2} , \tag{10}$$

where $a = \frac{\mu}{\mu_B}$ and μ_B is the Bohr Magneton. Now, following Weisskopf[2), we write the energy density of the vacuum electrons, w_m say, as follows

$$\begin{aligned} w_m &= -\frac{H'}{(2\pi)^2}\,\frac{mc^2}{\lambda_c^3} \sum_{n=0}^{\infty} \sum_{s=\pm 1} \int_{-\infty}^{\infty} \mathcal{E}_{n,s}(p)\, dp \\ &\equiv w_m^{(0)} + \Delta w_m^{(0)} \end{aligned} \tag{11}$$

where $w_m^{(0)}$ refers to the energy density when the AMM is neglected. To facilitate the evaluation w_m, we expand w_m to relative order α^2 and define

$$\Delta w_m^{(0)} \equiv \alpha\,\Delta w_m^{(1)} + \alpha^2\,\Delta w_m^{(2)} \tag{12}$$

It turns out, as we shall see shortly, that the contribution from $\Delta w_m^{(1)}$ to the energy density is divergent and proportional to H^2; this term is included in the unperturbed field energy[3] (renormalization of charge or field intensity). Consequently, the first non-linear contribution from Equation (12) is already proportional to α^2 which allows us to restrict ourselves to the lowest order term in "a" viz. the Schwinger result

$$a = \frac{\alpha}{2\pi} . \tag{13}$$

Thus, to the order required,

$$\mathcal{E}_{n,s}(P) = -\left\{P^2 + \left[\left\{1 + (2n+s+1)H^*\right\}^{1/2} + s\,\frac{\alpha}{4\pi}H^*\right]^2\right\}^{1/2} . \tag{14}$$

Expanding this expression to order α^2 we get

$$\begin{aligned} \mathcal{E}_{n,s}(P) = {} & \mathcal{E}^{(0)}_{n,s}(P) + \frac{\alpha}{4\pi}\, s H^* \left(\mathcal{E}^{(0)}_{n,s}(0)/\mathcal{E}^{(0)}_{n,s}(P)\right) \\ & - \frac{1}{8}\left(\frac{\alpha}{2\pi}\right)^2 H^{*2}\left(\mathcal{E}^{(0)}_{n,s}(P)\right)^{-1}\left\{\left(\mathcal{E}^{(0)}_{n,s}(0)/\mathcal{E}^{(0)}_{n,s}(P)\right)^2 - 1\right\} . \end{aligned} \tag{15}$$

The first term in this expression is the one used by the authors of reference 1-2), and we will not be concerned about it. To calculate $\Delta w_m^{(1)}$ we substitute the $\frac{\alpha}{4\pi}$ term from Equation (15) into Equation (11). Using the fact that

$$\mathcal{E}^{(0)}_{n,-1}(P) = \mathcal{E}^{(0)}_{(n+1),1}(P) \tag{16}$$

it is easily shown that

$$\sum_{n=0}^{\infty} \sum_{s=\pm 1} s \left(\mathcal{E}_{n,s}^{(0)}(0) / \mathcal{E}_{n,s}^{(0)}(p) \right) = - \frac{1}{\sqrt{1+p^2}} \quad . \tag{17}$$

Thus it follows that

$$\Delta w_m^{(1)} \propto H^2 \int_{-\infty}^{\infty} \frac{dp}{\sqrt{1+p^2}} \quad , \tag{18}$$

which justifies our previous statement following Equation (12).

Let us now consider the evaluation of $\Delta w_m^{(2)}$.

$$\begin{aligned} \Delta w^{(2)} &= \frac{H^{*3}}{8(2\pi)^4} \frac{mc^2}{\lambda_c^3} \sum_{n=0}^{\infty} \sum_{s=\pm 1} \int_{-\infty}^{\infty} \left(\mathcal{E}_{n,s}^{(0)}(p) \right)^{-1} \left\{ \left(\mathcal{E}_{n,s}^{(0)}(0) / \mathcal{E}_{n,s}^{(0)}(p) \right)^2 - 1 \right\} dp \\ &\equiv \frac{H^{*3}}{8(2\pi)^4} \frac{mc^2}{\lambda_c^3} \int_{-\infty}^{\infty} A(p)\, dp \end{aligned} \tag{19}$$

Making use of Equation (16) again we get

$$\begin{aligned} A(p) &= \sum_{n=0}^{\infty} \sum_{s=\pm 1} \left(\mathcal{E}_{n,s}^{(0)}(p) \right)^{-1} \left\{ \left(\mathcal{E}_{n,s}^{(0)}(0) / \mathcal{E}_{n,s}^{(0)}(p) \right)^2 - 1 \right\} \\ &= \left(\sum_{n=0}^{\infty} + \sum_{n=1}^{\infty} \right) \mathcal{E}_n^{-1}(p) \left\{ \frac{\mathcal{E}_n^2(0)}{\mathcal{E}_n^2(p)} - 1 \right\} \end{aligned} \tag{20}$$

where

$$\mathcal{E}_n(p) \equiv \sqrt{1+p^2+2n\, H^*} . \tag{21}$$

Now

$$\mathcal{E}_n^{-1}(p) \left\{ \left(\mathcal{E}_n^2(0) / \mathcal{E}_n^2(p) \right) - 1 \right\} = - \frac{p^2}{(1+p^2+x)^{3/2}} \tag{22}$$

where

$$x \equiv 2n\, H^* \tag{23}$$

It is thus obvious that $A(p)$ can be put in the following more transparent form

$$A(p) = F(0) + 2\sum_{n=1}^{\infty} F(x) , \tag{24}$$

to which the Euler summation formula (valid for any F) may be applied. Now the latter formula may be written as

$$F(0) + 2\sum_{n=1}^{\infty} F(b n) = \frac{2}{b}\left\{ \int_0^{\infty} F(z)\, dz + \sum_{t=1}^{\infty} \frac{B_t\, b^{2t}}{(2t)!}\left[F^{(2t-1)}(\infty) - F^{(2t-1)}(0)\right] \right\} , \tag{25}$$

where B_t are the Bernoulli numbers and $F^{(2t-1)}$ denotes the $(2t-1)^{th}$ derivative with respect to z where z = bn. In our particular case $X \equiv 2n\frac{H}{H_c}$ is equal to bn and hence b is equal to $2H^*$. Making use of the fact that

$$\frac{d^{(2t-1)}}{dx^{(2t-1)}}\left(p^2+1+x\right)^{-3/2} = -\frac{2}{\sqrt{\pi}}\,\Gamma(2t+1/2)\left(p^2+1+x\right)^{-1/2-2t} ,$$

it follows that

$$A(p) = -\frac{1}{H^*}\left\{ \int_0^{\infty} \frac{p^2\, dx}{(1+p^2+X)^{3/2}} + \frac{2}{\sqrt{\pi}} \sum_{t=1}^{\infty} \frac{B_t\, b^{2t}}{(2t)!}\,\Gamma(2t+1/2)\,\frac{p^2}{(p^2+1)^{2t+1/2}} \right\} . \tag{26}$$

Now the integral in Equation (26) is independent of H and thus when $A(p)$ is substituted in Equation (19) the result for $\Delta w^{(2)}$ contains this integral multiplied by a factor proportional to H^2 which, as before, is included in the unperturbed field energy and hence neglected in our subsequent discussion. Making use of the formula

$$\int_{-\infty}^{\infty} \frac{p^2 \, dp}{(p^2+1)^q} = \frac{1}{2}\sqrt{\pi} \quad \frac{\Gamma(q-3/2)}{\Gamma(q)} \quad \text{for } q > 1 \;, \tag{27}$$

we see

$$\int_{-\infty}^{\infty} A(p)\, dp = -\frac{1}{H'^2} \sum_{t=1}^{\infty} \frac{B_t \, \ell^{2t}}{(2t)!} \; \Gamma(2t-1) \;. \tag{28}$$

Now making use of the fact that

$$\Gamma(2t-1) = \int_0^{\infty} d\eta \; e^{-\eta} \; \eta^{2t-2} \;, \tag{29}$$

substituting Equation (28) in Equation (19), and replacing t by n we get

$$\Delta w_m^{(2)} = -\frac{1}{8} \frac{H'^2}{(2\pi)^4} \frac{mc^2}{\lambda_c^3} \sum_{n=1}^{\infty} \int_0^{\infty} d\eta \, e^{-\eta} \, \eta^{2n-2} \; \frac{B_n \, \ell^{2n}}{(2n)!} \;. \tag{30}$$

Making use of the formula

$$\coth x - \frac{1}{x} = \sum_{n=1}^{\infty} \frac{2^{2n} B_n}{(2n)!} \; x^{2n-1} \;, \tag{31}$$

we finally get

$$L_2 = -\alpha^2 \Delta w_m^{(2)} = \frac{1}{32\pi^2}\left(\frac{\alpha}{2\pi}\right)^2 \frac{mc^2}{\lambda_c^3} H'^2 \int_0^{\infty} \frac{d\eta}{\eta^2} e^{-\eta} \left\{ \eta H' \coth(\eta H') - 1 \right\} . \tag{32}$$

This equation for L_2 represents the desired correction to the non-linear Lagrangian of the magnetic field which arises from the fact that the electron has an anomalous magnetic moment.

It is now of interest to consider the weak and strong fields limits. We find ($H^* = \frac{H}{H_c}$)

$$L_2 = \frac{1}{96\pi^2}\left(\frac{\alpha}{2\pi}\right)^2 \frac{mc^2}{\lambda_c^3}\left(\frac{H}{H_c}\right)^4 \quad \text{for } H \ll H_c \,, \tag{33}$$

$$L_2 = \frac{1}{32\pi^2}\left(\frac{\alpha}{2\pi}\right)^2 \frac{mc^2}{\lambda_c^3}\left(\frac{H}{H_c}\right)^2 \ln\left(\frac{H}{H_c}\right) \quad \text{for } H \gg H_c \,. \tag{34}$$

APPENDIX A: CALCULATION OF THE ENERGY EIGENVALUES OF A RELATIVISTIC ELECTRON WITH AN ANOMALOUS MAGNETIC MOMENT IN A CONSTANT MAGNETIC FIELD OF ARBITRARY STRENGTH

As discussed in the main text we use the usual Dirac equation, with the addition of the Pauli anomalous moment interaction term (see Equation (9)), which we repeat for convenience

$$i\,\frac{\partial \psi}{\partial t} = \left\{ \underset{\sim}{\alpha}\cdot(\underset{\sim}{p} + e\underset{\sim}{A}) + \gamma_4 m + \mu \gamma_4 \; \underset{\sim}{\Sigma}\cdot\underset{\sim}{H} \right\} \psi \; . \qquad \text{(A1)}$$

It is also convenient to set $\hbar = c = 1$ for the purpose of this derivation. We now choose the following representation for the γ matrices

$$\gamma_i = -i\gamma_4\,\alpha_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix} \; ; \qquad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix};$$

$$\gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; , \qquad \Sigma_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \; ,$$

σ_i being the usual Pauli matrices, and we set

$$\pi_i = -i\,\nabla_i + e A_i \; . \qquad \text{(A2)}$$

The magnetic field is considered to be directed along the z axis and it is convenient to use a cylindrical coordinate system[7]. Similar to the procedure used by Johnson and

Lippmann[5] in the case $\mu = 0$, we write the solution to Equation (A1) in the form

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} , \qquad \text{(A3)}$$

and we set

$$\psi_{1,3} = \frac{1}{\sqrt{2\pi}} \, e^{i p_z z} \, e^{i(\ell-1)\phi} \, f_{1,3}(r) , \qquad \text{(A4)}$$

$$\psi_{2,4} = \frac{1}{\sqrt{2\pi}} \, e^{i p_z z} \, e^{i\ell\phi} \, f_{2,4}(r) , \qquad \text{(A5)}$$

where p_z is the momentum of the electron in the z-direction and $f_{1,2,3,4}(r)$ are the functions of r which are to be determined. Using the fact that

$$\pi_x \pm i\pi_y = -i \, e^{\pm i\phi} \left(\frac{\partial}{\partial r} \pm \frac{i}{r} \frac{\partial}{\partial \phi} \mp \gamma r \right) , \qquad \text{(A6)}$$

and

$$\gamma_4 \Sigma_3 \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ -\psi_2 \\ -\psi_3 \\ \psi_4 \end{pmatrix} , \qquad \text{(A7)}$$

it follows from the above equations that

$$(E - m_1) f_1(r) = -i\left[\frac{\partial}{\partial r} + \frac{\ell}{r} + \gamma r\right] f_4(r) + P_z f_3(r) , \quad \text{(A8)}$$

$$(E - m_2) f_2(r) = -i\left[\frac{\partial}{\partial r} - \frac{(\ell-1)}{r} - \gamma r\right] f_3(r) - P_z f_4(r) , \quad \text{(A9)}$$

$$(E - m_1) f_3(r) = -i\left[\frac{\partial}{\partial r} + \frac{\ell}{r} + \gamma r\right] f_2(r) + P_z f_1(r) , \quad \text{(A10)}$$

$$(E - m_2) f_4(r) = -i\left[\frac{\partial}{\partial r} - \frac{(\ell-1)}{r} - \gamma r\right] f_1(r) - P_z f_2(r) , \quad \text{(A11)}$$

where

$$\gamma = \frac{eH}{2} = \frac{1}{2} m^2\left(\frac{H}{H_c}\right) ;$$

$$m_1 = m - \mu H ; \; m_2 = m + \mu H ; \; \mu = a \mu_B . \quad \text{(A12)}$$

As before μ_B denotes the Bohr Magneton.

It is now convenient to set $\rho = \gamma r^2$. As a result Equations (A9) and (A11) may be re-written as

$$\left[\rho \frac{\partial^2}{\partial \rho^2} + \frac{\partial}{\partial \rho} - \frac{\ell^2}{4\rho} - \frac{\rho}{4} - \frac{(\ell-1)}{2} + B_1\right] f_4(\rho) + B_3 f_2(\rho) = 0 , \quad \text{(A13)}$$

$$\left[\rho \frac{\partial^2}{\partial \rho^2} + \frac{\partial}{\partial \rho} - \frac{\ell^2}{4\rho} - \frac{\rho}{4} - \frac{(\ell-1)}{2} + B_2\right] f_2(\rho) - B_3 f_4(\rho) = 0 . \quad \text{(A14)}$$

where

$$B_1 = \frac{1}{4\gamma}\left[E^2 - m_1 m_2 - P_z^2 + DE\right] \quad , \qquad \text{(A15)}$$

$$B_2 = \frac{1}{4\gamma}\left[E^2 - m_1 m_2 - P_z^2 - DE\right] \quad , \qquad \text{(A16)}$$

$$B_3 = \frac{P_z}{4\gamma} \quad , \qquad \text{(A17)}$$

and

$$D = m_2 - m_1 = 2\mu H \quad . \qquad \text{(A18)}$$

The functions f_1 and f_3 obey similar equations but these will not concern us here. To solve these equations we use the familiar power series method. Assume

$$f_4(\rho) = e^{-\frac{\rho}{2}} \rho^s \sum_{N=0}^{\infty} C_N \rho^N \quad , \qquad \text{(A19)}$$

$$f_2(\rho) = e^{-\frac{\rho}{2}} \rho^s \sum_{N=0}^{\infty} D_N \rho^N \quad . \qquad \text{(A20)}$$

We now substitute Equations (A19) and (A20) into Equations (A13) and (A14). Equating the coefficients of $e^{-\frac{\rho}{2}} \rho^{N+S-1}$, we find

$$C_{N-1}\left[B_1 - N + \tfrac{1}{2} - \frac{(\ell-1)}{2} - S\right] + C_N\left[(N+S)^2 - \left(\tfrac{\ell}{2}\right)^2\right] + B_3 D_{N-1} = 0 , \tag{A21}$$

$$D_{N-1}\left[B_2 - N + \tfrac{1}{2} - \frac{(\ell-1)}{2} - S\right] + D_N\left[(N+S)^2 - \left(\tfrac{\ell}{2}\right)^2\right] - B_3 C_{N-1} = 0 . \tag{A22}$$

Taking $N=0$, it immediately follows that $2S=\pm\ell$. However, to avoid divergence difficulties for $\rho=0$ it is necessary to discard the $2S=-\ell$ solution. Thus, from henceforth, we take $2S=\ell$. To obtain a well-behaved wave-function we assume, as is usual, that our series terminates at $N=N'$ i.e.

$$C_{N'+1} = 0 , \quad D_{N'+1} = 0 .$$

Now let $N=N'+1$ in Equations (A21) and (A22) and we get

$$C_{N'}\left[B_1 - N' - \ell\right] + B_3 D_{N'} = 0 , \tag{A23}$$

$$C_{N'} B_3 - D_{N'}\left[B_2 - N' - \ell\right] = 0 . \tag{A24}$$

It immediately follows that

$$\left[B_1 - (N'+\ell)\right]\left[B_2 - (N'+\ell)\right] + B_3^2 = 0 . \tag{A25}$$

This equation can now be solved for E with the result

$$E = \pm m\left\{\left(\frac{P_3}{m}\right)^2 + \left[\left\{1 + 2\frac{H}{H_c}(N'+\ell)\right\}^{1/2} \pm \tfrac{1}{2} a \frac{H}{H_c}\right]^2\right\}^{1/2} , \tag{A26}$$

where $N' + \ell = 0, 1, 2, \cdots$.

It is now convenient to re-write this equation in the form

$$E_{n,s,\xi}(p) = \pm m \left\{ \left(\frac{p_z}{m}\right)^2 + \left[\left\{ 1 + \frac{H}{H_c}(2n+s+1) \right\}^{1/2} + \frac{1}{2}\,\xi\, a \frac{H}{H_c} \right]^2 \right\}^{1/2} \quad \text{(A27)}$$

where $s = \pm 1$ and $\xi = \pm 1$. Going now to the non-relativistic limit leads to the result that

$$E^{(NR)}_{n,s,\xi}(p) = \pm m \left\{ 1 + \frac{1}{2}\left(\frac{p_z}{m}\right)^2 + \frac{H}{H_c}\left[\left(n + \tfrac{1}{2}\right) + \frac{s}{2}\left(1 + a\,\frac{\xi}{s}\right) \right] \right\}, \quad \text{(A28)}$$

and thus it follows that

$$\xi = s \quad . \quad \text{(A29)}$$

Thus we get our final expression (reinserting the c)

$$E_{n,s}(p) = \pm mc^2 \left\{ \left(\frac{p_z}{mc}\right)^2 + \left[\left\{ 1 + \frac{H}{H_c}(2n+s+1) \right\}^{1/2} + \frac{1}{2}\, s\, a \frac{H}{H_c} \right]^2 \right\}^{1/2} \quad \text{(A30)}$$

which corresponds to Equation (10) in the main text.

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